Renormalization group theory of nematic ordering in *d*-wave superconductors

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We examine the quantum theory of the spontaneous breaking of lattice rotation symmetry in *d*-wave superconductors on the square lattice. This is described by a field theory of an Ising nematic order parameter coupled to the gapless fermionic quasiparticles. We determine the structure of the renormalization group to all orders in a $1/N_f$ expansion, where N_f is the number of fermion spin components. Asymptotically exact results are obtained for the quantum critical theory in which, as in the large N_f theory, the nematic order has a large anomalous dimension, and the fermion spectral functions are highly anisotropic.

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I. INTRODUCTION

There has been a great deal of research on the onset of a variety of competing orders in the hole-doped cuprate superconductors. In Ref. 1, a classification of spin-singlet order parameters at zero momentum was presented: such orders are able to couple efficiently to the gapless nodal quasiparticle excitations of a *d*-wave superconductor. Our focus in the present paper will be on one such order parameter: "nematic" ordering in which the square lattice (tetragonal) symmetry of the *d*-wave superconductor is spontaneously reduced to rectangular (orthorhombic) symmetry.[2] This transition is characterized by an Ising order parameter, but the quantum phase transition is not in the usual Ising universality class[1] because the coupling to the gapless fermionic quasiparticles changes the nature of the quantum critical fluctuations. Our work is motivated by recent neutron-scattering observations[3] of a strongly temperature (T)-dependent susceptibility to nematic ordering in detwinned crystals of $YBa_2Cu_3O_{6.45}$. On a more microscopic scale, the scanning tunneling microscopy observations of Kohsaka *et al.*[4] show bond-centered modulations in the local density of states which also break the rotational symmetry of the square lattice. These observations indicate that such square lattice space-group symmetry-breaking states are a fundamental ingredient in the physics of the enigmatic "pseudogap" region of the cuprates, and we will focus on the simplest underlying nematic order parameter. In reality, this nematic order is likely accompanied by density-wave modulations at a wave vector of a few lattice spacings, but it is plausible[5,6] that it is the nematic correlation length which diverges first. We also refer the reader to Ref. 7 for further discussion on the physical importance and experimental relevance of nematic ordering in the cuprate superconductors.

An initial renormalization group (RG) analysis[1] of nematic ordering at $T=0$ found runaway flow to strong coupling in a computation based on an expansion in $(3-d)$, where d is the spatial dimensionality. More recently, Kim *et al.*[7] argued that a second-order quantum phase transition existed in the limit of large N_f, where N_f is the number of spin components (the physical case corresponding to $N_f=2$). Here we will present a RG analysis using the framework of the $1/N_f$ expansion. We will find that a fixed point does indeed exist at order $1/N_f$, describing a second-order quantum transition associated with the onset of long-range nematic order. However, the scaling properties near the fixed point have a number of unusual properties, controlled by the marginal flow of a "dangerously" irrelevant parameter. This parameter is v_Δ/v_F, where v_Δ and v_F are the velocities of the nodal fermions parallel and perpendicular to the Fermi surface (see Fig. 1). We will show that the fixed point has $(v_\Delta/v_F)^*=0$ and so the transition is described by an "infinite anisotropy" away from the "relativistic" fixed points found for other competing orders.[1] It is important to note, however, that even though the fixed point has $(v_\Delta/v_F)^*=0$, it is not described by an effectively one-dimensional theory of a straight Fermi surface; a fully two-dimensional theory is needed as is discussed in more detail in Sec. IV. The approach to this fixed point is logarithmically slow, and the physical properties have to be computed at finite v_Δ/v_F. Our main results for the RG flow of the velocities are in Eqs. (3.8) and (3.9), where $C_{1,2,3}$ are functions only of v_Δ/v_F which are specified in Eq. (3.18); for $v_\Delta/v_F \ll 1$, the equations reduce to the explicit forms in Eqs. (3.22)–(3.24). These (and related) equations can be integrated in a standard manner to yield the dependence of observables on temperature and deviation from the nematic critical point, and the results are in Figs. 6–8.

The existence of a fixed point at $(v_\Delta/v_F)^*=0$ suggests that we analyze the theory directly in the limit of small v_Δ/v_F without an appeal to an expansion in powers of $1/N_f$. The nature of the small v_Δ/v_F limit is quite subtle and has to be taken with care: it will be described in Sec. IV. We argue there that the fluctuations of the nematic order are controlled by a small parameter which is $\sim v_\Delta/(N_f v_F)$ and not $1/N_f$ alone. Consequently, we believe our computations are controlled in the limit of small v_Δ/v_F even for $N_f=2$. Indeed, the

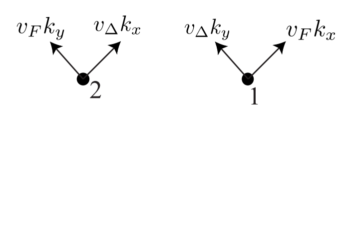

FIG. 1. Nodal points of the *d*-wave superconductor in the square lattice Brillouin zone. The two-component $\Psi_{1,2}$ fermions are in the vicinity of the labeled nodal points and their partners at diagonally opposite points.

results in Eqs. (3.22)–(3.24), and many other related results, are expected to be exact as we approach the quantum critical point.

The outline of the remainder of this paper is as follows. The field theory for the nematic ordering transition will be reviewed in Sec. II, along with a discussion of the $1/N_f$ expansion. The RG analysis to order $1/N_f$ will be presented in Sec. III. Finally, higher-order corrections in $1/N_f$ and the nature of the small v_Δ/v_F limit will be discussed in Sec. IV.

II. FIELD THEORY

We begin by reviewing the field theory for the nematic ordering transition introduced in Ref. 1. The action for the field theory, S, has three components,

$$S = S_\Psi + S_\phi^0 + S_{\Psi\phi}. \tag{2.1}$$

The first term in the action, S_Ψ, is simply for the low-energy fermionic excitations in the $d_{x^2-y^2}$ superconductor. We begin with the electron annihilation operator $c_{\mathbf{q}a}$ at momentum $\mathbf{q}$ and spin $a=\uparrow,\downarrow$. We will shortly generalize the theory to one in which $a=1,\ldots,N_f$, with N_f an arbitrary integer. We denote the $c_{\mathbf{q}a}$ operators in the vicinity of the four nodal points $\mathbf{q}=(\pm K,\pm K)$ by (anticlockwise) f_{1a}, f_{2a}, f_{3a}, and f_{4a}. Now introduce the two-component Nambu spinors $\Psi_{1a}=(f_{1a},\varepsilon_{ab}f_{3b}^\dagger)$ and $\Psi_{2a}=(f_{2a},\varepsilon_{ab}f_{4b}^\dagger)$, where $\varepsilon_{ab}=-\varepsilon_{ba}$ and $\varepsilon_{\uparrow\downarrow}=1$ [we will follow the convention of writing out spin indices (a,b) explicitly, while indices in Nambu space will be implicit]. Expanding to linear order in gradients from the nodal points, the Bogoliubov action for the fermionic excitations of a d-wave superconductors can be written as (see Fig. 1)

$$S_\Psi = \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \sum_{a=1}^{N_f} \Psi_{1a}^\dagger(-i\omega_n + v_F k_x \tau^z + v_\Delta k_y \tau^x)\Psi_{1a} + \int \frac{d^2k}{(2\pi)^2} T \sum_{\omega_n} \sum_{a=1}^{N_f} \Psi_{2a}^\dagger(-i\omega_n + v_F k_y \tau^z + v_\Delta k_x \tau^x)\Psi_{2a}. \tag{2.2}$$

Here ω_n is a Matsubara frequency, τ^α are the Pauli matrices which act in the fermionic particle-hole space, $k_{x,y}$ measure the wave vector from the nodal points and have been rotated by 45° from $q_{x,y}$ coordinates, and v_F and v_Δ are the velocities. The sum over a in Eq. (2.2) can be considered to extend over an arbitrary number N_f.

The second term, S_ϕ^0, describes the effective action for the Ising nematic order parameter, which we represent as a real field ϕ. Considering only the contribution to its action generated by high-energy electronic excitations, we obtain only the analytic terms present in the quantum Ising model,

$$S_\phi^0 = \int d^2x d\tau \left[\frac{1}{2}(\partial_\tau\phi)^2 + \frac{c^2}{2}(\nabla\phi)^2 + \frac{r}{2}\phi^2 + \frac{u_0}{24}\phi^4\right]. \tag{2.3}$$

Here τ is imaginary time, c is a velocity, r tunes the system across the quantum critical point, and u_0 is a quartic self-interaction.

The final term in the action, $S_{\Psi\phi}$, couples the Ising nematic order, ϕ, to the nodal fermions, Ψ_{1a} and Ψ_{2a}. This can be deduced by a symmetry analysis,[1] and the important term is a trilinear "Yukawa" coupling,

$$S_{\Psi\phi} = \int d^2x d\tau \left[\lambda_0 \phi \sum_{a=1}^{N_f} (\Psi_{1a}^\dagger \tau^x \Psi_{1a} + \Psi_{2a}^\dagger \tau^x \Psi_{2a})\right], \tag{2.4}$$

where λ_0 is a coupling constant.

We can now see that the action S describes the couplings between the ϕ bosons and the Ψ fermions, both of which have a "relativistic" dispersion spectrum. However, the velocity of "light" in their dispersion spectra, c, v_F, and v_Δ, are not all equal. If we choose equal velocities with $v_F=v_\Delta=c$, then the decoupled theory $S_\Psi+S_\phi^0$ does have a relativistically invariant form. However, even in this case, the fermion-boson coupling $S_{\Psi\phi}$ is *not* relativistically invariant: the coupling matrix τ^x in Eq. (2.4) breaks Lorentz symmetry, and this feature will be crucial for our analysis. If we replace τ^x by τ^y in Eq. (2.4), then we obtain an interacting relativistically invariant theory for the equal-velocity case, which was studied in Ref. 1 as a description of the transition from a $d_{x^2-y^2}$ to a $d_{x^2-y^2}+id_{xy}$ superconductor.

A renormalization group analysis of the above action has been presented previously[1] in an expansion in $(3-d)$. No fixed point describing the onset of nematic order was found with the couplings u_0 and λ_0 flowing away to infinity. This runaway flow was closely connected to the non-Lorentz-invariant structure of $S_{\Psi\phi}$ and the resulting flow of the velocities away from each other. In contrast, for the transition to a $d_{x^2-y^2}+id_{xy}$ superconductor, a stable fixed point was found at which the velocities flowed to equal values at long scales.[1] Similar relativistically invariant fixed points are also found in other cases involving the onset of spin or charge-density-wave orders at wave vectors which nest the separation between nodal points.[8,9]

A. Large N_f expansion

We will analyze S in the context of the $1/N_f$ expansion. Formally, this involves integrating out the $\Psi_{1,2}$ fermions and obtaining the resulting nonlocal effective action for ϕ. Near the potential quantum critical point, the nonlocal terms so generated are more important in the infrared than the terms S_ϕ^0 in Eq. (2.3), apart from the "mass" term r. So we can drop the remaining terms in S_ϕ^0; also for convenience we rescale $\phi\to\phi/\lambda_0$ and $r\to N_f r \lambda_0^2$ and obtain the local-field theory,

$$S = S_\Psi + \int d^2x d\tau \left[\frac{N_f r}{2}\phi^2 + \phi \sum_{a=1}^{N_f} (\Psi_{1a}^\dagger \tau^x \Psi_{1a} + \Psi_{2a}^\dagger \tau^x \Psi_{2a})\right]. \tag{2.5}$$

This local-field theory will form the basis of all our RG analysis. Note that this field theory has only three parameters: r, v_F, and v_Δ. So any RG equations can be expressed only in terms of these parameters, as will be presented in Sec. III.

The large N_f expansion proceeds by integrating out the Ψ fermions, yielding an effective action S_ϕ for the nematic or-

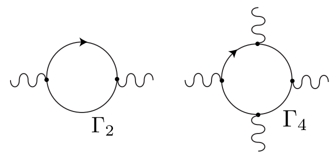

FIG. 2. Feynman graph expansion for the effective action S_ϕ. The full lines are fermion propagators, while the wavy lines are ϕ insertions.

der ϕ. It is important to note that S_ϕ is a complicated nonlocal functional of ϕ, which however depends only on r, v_F, and v_Δ. Also, in writing out explicit forms for S_ϕ it is essential to keep subtle issues on orders of limits in mind. In particular, for the large N_f phase diagram, we need the effective potential for ϕ at zero k and ω. Thus we need an expansion of the effective potential in the regime $|\phi| \gg |k|, |\omega|$—the structure of this was discussed in Ref. 7 and yielded a second-order transition in the limit of large N_f. In contrast, for our RG analysis here, we will work in the quantum critical region, where $\langle\phi\rangle = 0$, and so we need the functional for $|\phi| \ll |k|, |\omega|$. In this case, we can expand S_ϕ in powers of ϕ to yield the formal result,

$$\frac{S_\phi}{N_f} = \frac{1}{2}\int_K (r + \Gamma_2(K))|\phi(K)|^2 + \frac{1}{4}\prod_{i=1}^{4}\int_{K_i} \delta\left(\sum_i K_i\right) \times \Gamma_4(K_1,K_2,K_3,K_4)\phi(K_1)\phi(K_2)\phi(K_3)\phi(K_4) + \cdots . \tag{2.6}$$

Here the $K_i \equiv (k_i, \omega_i)$ are three momenta. The functions Γ_i are all given by one-fermion loop Feynman graphs with i insertions of the external ϕ vertices, as shown in Fig. 2; we denote the external momenta of these vertices, K_i, clockwise around the fermion loop. The Feynman loop integrals are quite tedious to evaluate, especially for large i, and so below we only present explicit expressions for the needed low-order terms. However, all the Γ_i are universal functions of only the momenta and v_F and v_Δ

Our analysis will require the explicit form of $\Gamma_2(K)$. This can be written as

$$\Gamma_2(K) = \Pi_2(k_x,k_y,\omega) + \Pi_2(k_y,k_x,\omega), \tag{2.7}$$

with the terms representing the contributions of the Ψ_1 and Ψ_2 fermions, respectively. The one-fermion loop diagram yields

$$\Pi_2(k_x,k_y,\omega) = \int \frac{d^2p}{4\pi^2}\int \frac{d\Omega}{2\pi}\mathrm{Tr}\{\tau^x[-i(\Omega+\omega) + v_F(p_x+k_x)\tau^z + v_\Delta(p_y+k_y)\tau^x]^{-1} \times \tau^x(-i\Omega + v_F p_x\tau^z + v_\Delta p_y \tau^x)^{-1}\} = \frac{1}{16 v_F v_\Delta}\frac{(\omega^2 + v_F^2 k_x^2)}{(\omega^2 + v_F^2k_x^2 + v_\Delta^2 k_y^2)^{1/2}}. \tag{2.8}$$

Here, we have subtracted out a constant which shifts the position of the critical point.

For Γ_4 we will only need the cases where two of the four external three momenta vanish. The first is

$$\Gamma_4(K,-K,0,0) = \Pi_{4a}(k_x,k_y,\omega) + \Pi_{4a}(k_y,k_x,\omega), \tag{2.9}$$

where

$$\Pi_{4a}(k_x,k_y,\omega) = \int \frac{d^2p}{4\pi^2}\int \frac{d\Omega}{2\pi}\mathrm{Tr}[\{\tau^x[-i(\Omega+\omega) + v_F(p_x + k_x)\tau^z + v_\Delta(p_y+k_y)\tau^x]^{-1}\}^3 \times \tau^x(-i\Omega + v_F p_x \tau^z + v_\Delta p_y\tau^x)^{-1}] = \frac{1}{2v_\Delta^2}\frac{\partial^2\Pi_2(k_x,k_y,\omega)}{\partial k_y^2} = -\frac{1}{32 v_F v_\Delta}\frac{(\omega^2 + v_F^2k_x^2)(\omega^2 + v_F^2k_x^2 - 2v_\Delta^2k_y^2)}{(\omega^2 + v_F^2k_x^2 + v_\Delta^2k_y^2)^{5/2}}. \tag{2.10}$$

The other case with two vanishing three momenta is

$$\Gamma_4(K,0,-K,0) = \Pi_{4b}(k_x,k_y,\omega) + \Pi_{4b}(k_y,k_x,\omega), \tag{2.11}$$

where

$$\Pi_{4b}(k_x,k_y,\omega) = \int \frac{d^2p}{4\pi^2}\int \frac{d\Omega}{2\pi}\mathrm{Tr}[\{\tau^x[-i(\Omega+\omega) + v_F(p_x+k_x)\tau^z + v_\Delta(p_y+k_y)\tau^x]^{-1}\}^2 \times [\tau^x(-i\Omega + v_F p_x\tau^z + v_\Delta p_y \tau^x)^{-1}]^2] = \frac{1}{16 v_F v_\Delta}\frac{(\omega^2 + v_F^2k_x^2)(\omega^2 + v_F^2k_x^2 - 2v_\Delta^2k_y^2)}{(\omega^2 + v_F^2k_x^2 + v_\Delta^2k_y^2)^{5/2}}. \tag{2.12}$$

From the effective action S_ϕ in Eq. (2.6), we can obtain the $1/N_f$ corrections to various observable quantities.

For the nematic susceptibility, χ_ϕ, we have

$$\chi_\phi^{-1} = r + \frac{1}{N_f}\int_K \frac{2\Gamma_4(K,-K,0,0) + \Gamma_4(K,0,-K,0)}{\Gamma_2(K) + r}. \tag{2.13}$$

Using the values in Eqs. (2.10) and (2.12), we observe that the $1/N_f$ correction is identically zero. This can be traced to the arguments in Ref. 7 (based on gauge invariance and the fact that the coupling in Eq. (2.4) is to a globally conserved fermion current) that the effective potential for the field ϕ is unrenormalized by the low-energy fermion action. Indeed, this argument implies that all higher-order terms in $1/N_f$ also vanish and that $\chi_\phi^{-1} = r$ exactly in the present continuum field theory. So we may conclude that the susceptibility exponent, γ, for the nematic order parameter has the exact value,

$$\gamma = 1. \tag{2.14}$$

Note that there are $1/N_f$ corrections to the momentum and frequency dependence of the nematic susceptibility, and these will appear in our RG analysis below.

Turning to the $1/N_f$ expansion for the Ψ_1 Green's function, we write this as

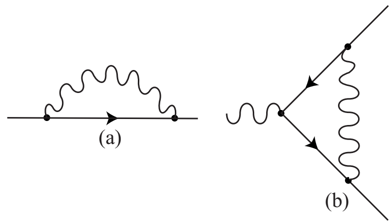

FIG. 3. Order $1/N_f$ contributions to the (a) self-energy Σ_1 of Ψ_1 fermions and (b) the vertex Ξ.

$$G_{\Psi_1}^{-1}(k,\omega) = -i\omega + v_F k_x \tau^z + v_\Delta k_y \tau^x - \Sigma_1(k,\omega). \quad (2.15)$$

The self-energy is given by the Feynman graph in Fig. 3,

$$\Sigma_1(k,\omega) = \frac{1}{N_f}\int \frac{d^2p}{4\pi^2}\int \frac{d\Omega}{2\pi} \times \frac{[i(\Omega+\omega) - v_F(p_x+k_x)\tau^z + v_\Delta(p_y+k_y)\tau^x]}{(\Omega+\omega)^2 + v_F^2(p_x+k_x)^2 + v_\Delta^2(p_y+k_y)^2} \times \frac{1}{\Gamma_2(p,\Omega)+r}. \quad (2.16)$$

Finally, we will also need the $1/N_f$ correction to the boson-fermion vertex (the Yukawa coupling) in Eq. (2.4). At zero external momenta and frequencies, this vertex is renormalized by Fig. 3 to

$$\Xi = \tau^x + \frac{1}{N_f}\int \frac{d^2p}{4\pi^2}\int \frac{d\Omega}{2\pi}\Bigg[\tau^x(-i\Omega + v_F p_x \tau^z + v_\Delta p_y \tau^x)^{-1} \times \tau^x(-i\Omega + v_F p_x \tau^z + v_\Delta p_y \tau^x)^{-1}\tau^x \frac{1}{\Gamma_2(p,\Omega)+r}\Bigg]. \quad (2.17)$$

The RG equations will be obtained in Sec. III from Eqs. (2.13), (2.16), and (2.17).

III. RENORMALIZATION GROUP ANALYSIS

We begin our discussion of the RG by describing the general structure which applies to *all* orders in the $1/N_f$ expansion. The RG will be based on a consideration of the renormalization of the local-field theory S in Eq. (2.5). We are interested in the behavior of this field theory under the rescaling transformation,

$$k = k' e^{-\ell},$$

$$\omega = \omega' e^{-\ell}. \quad (3.1)$$

Note that we have not introduced a dynamic critical exponent z for the rescaling of the frequency. This is because we will allow both velocities v_F and v_Δ to flow, and this flow will effectively account for any anomalous dynamic scaling. Under this space-time rescaling, we also have a rescaling of the fermion and boson fields,

$$\Psi_{1,2}(k,\omega) = \Psi'_{1,2}(k',\omega')\exp\left(\frac{1}{2}\int_0^\ell d\ell(4-\eta_f)\right),$$

$$\phi(k,\omega) = \phi'(k',\omega')\exp\left(\frac{1}{2}\int_0^\ell d\ell(5-\eta_b)\right). \quad (3.2)$$

Here we have allowed the anomalous dimensions $\eta_{b,f}$ to be scale dependent, as that will be the case below. To implement these field rescalings, we have to determine how the field scales are defined. For the fermions, as is conventional, we set the scale of $\Psi_{1,2}$ so that the coefficients of the time derivative terms in S_Ψ remain unity. For the nematic order parameter, ϕ, there is no "kinetic-energy" term in Eq. (2.5), and so we cannot use the conventional method. Instead, recall that the Yukawa coupling λ_0 was absorbed into the overall scale of ϕ in Eq. (2.5). Therefore it is natural to set the scale of ϕ so that the boson-fermion vertex in Eq. (2.5) remains fixed at unity. This is also consonant with the fact that the boson kinetic energy comes entirely from the fermion loops.

The RG equations now follow from the low-frequency forms of the fermion self-energy Σ_1 and the vertex Ξ. These will have a dependence on an ultraviolet cutoff, Λ. We will discuss below an explicit method of applying this cutoff. For now, we note that for the RG we only need the logarithmic derivative of the self-energy and the vertex. Thus let us write

$$\Lambda\frac{d}{d\Lambda}\Sigma_1(k,\omega) = C_1(-i\omega) + C_2 v_F k_x \tau^z + C_3 v_\Delta k_y \tau^x,$$

$$\Lambda\frac{d}{d\Lambda}\Xi = C_4\tau^x. \quad (3.3)$$

On the right-hand side, we assume (as in the usual field-theoretic RG) that the limit $\Lambda\to\infty$ can be safely taken. Then a simple dimensional analysis shows that the C_{1-4} are all universal dimensionless functions of the velocity ratio v_Δ/v_F. In principle, the above method can be generalized to any needed order in the $1/N_f$ expansion by taking the appropriate logarithmic cutoff derivative of the self-energy Σ_1 and the vertex Ξ, as reviewed in Ref. 10.

With the results in Eq. (3.3), we can set the scaling dimensions of the fields. Considering the renormalization of the $-i\omega$ term in S_Ψ, we obtain

$$\eta_f = -C_1. \quad (3.4)$$

Imposing the unit value of the Yukawa coupling in Eq. (2.5) we obtain

$$\eta_b = 1 + 2C_4 - 2\eta_f = 1 + 2C_4 + 2C_1. \quad (3.5)$$

Note the large value of η_b in the large N_f theory. This is a consequence of the fact that the ϕ kinetic energy is tied to the nonanalytic fermion loop contribution. We can now also use the nonrenormalization of the ϕ^2 term in Eq. (2.5), as noted in Eq. (2.14), to determine the flow equation for the "mass" r. This defines the correlation length exponent ν by the RG equation,

$$\frac{dr}{d\ell} = \frac{1}{\nu}r, \quad (3.6)$$

for small r, and we have the non-mean-field value,

$$\nu = \frac{1}{2 - \eta_b}. \tag{3.7}$$

The results in Eq. (3.3) also yield the flow equations for the velocities. By considering the renormalization of the k-dependent terms in S_Ψ we determine that the velocity v_F flows according to

$$\frac{dv_F}{d\ell} = (-\eta_f - C_2)v_F = (C_1 - C_2)v_F, \tag{3.8}$$

while the velocity v_Δ obeys

$$\frac{dv_\Delta}{d\ell} = (-\eta_f - C_3)v_\Delta = (C_1 - C_3)v_\Delta. \tag{3.9}$$

Clearly, the ratios of the velocity obeys

$$\frac{d(v_\Delta/v_F)}{d\ell} = (C_2 - C_3)(v_\Delta/v_F). \tag{3.10}$$

All that remains now is to determine the C_{1-4} as a function of v_Δ/v_F. These functions also depend on the mass r, but we will henceforth restrict ourselves to the vicinity of the critical point by setting $r=0$. We reiterate that, in principle, our method allows us to obtain these functions to any needed order in the $1/N_f$ expansion. However, we will only obtain explicit results below to first order $1/N_f$.

The computation of C_{1-4} requires the imposition of a ultraviolet momentum cutoff. We implement this[1] by multiplying both the Fermi and Bose propagators by a smooth cutoff function $\mathcal{K}(k^2/\Lambda^2)$. Here $\mathcal{K}(y)$ is an arbitrary function with $\mathcal{K}(0)=1$ and which falls off rapidly with y at $y \sim 1$, e.g., $\mathcal{K}(y)=e^{-y}$. We will show below that the RG equations are independent of the particular choice of $\mathcal{K}(y)$.

Let us now discuss the evaluation of C_{1-3}. Schematically, we can write for the fermion self-energy

$$\Sigma_1(K) = \int \frac{d^3P}{8\pi^3} F(P+K)G(P)\mathcal{K}\left(\frac{(p+k)^2}{\Lambda^2}\right)\mathcal{K}\left(\frac{p^2}{\Lambda^2}\right), \tag{3.11}$$

where $K \equiv (k,\omega)$ and $P \equiv (p,\Omega)$ are three momenta, and F and G are functions which can be obtained by comparing this expression to Eq. (2.16). Also notice that (at $r=0$) F and G are both homogenous functions of three momenta of degree of -1. Expanding to first order in K_μ (μ is a space-time index), we have

$$\Sigma_1(K) \approx K_\mu \int \frac{d^3P}{8\pi^3}\left[\frac{\partial F(P)}{\partial P_\mu} G(P)\mathcal{K}^2\left(\frac{p^2}{\Lambda^2}\right) + F(P)G(P)\mathcal{K}\left(\frac{p^2}{\Lambda^2}\right)\frac{2p_\mu}{\Lambda^2}\mathcal{K}'\left(\frac{p^2}{\Lambda^2}\right)\right], \tag{3.12}$$

where p_μ is the three vector $(0,p_x,p_y)$ [while $P_\mu = (\Omega,p_x,p_y)$]. Now taking the Λ derivative

$$\Lambda\frac{d}{d\Lambda}\Sigma_1(K) \approx K_\mu \int \frac{d^3P}{8\pi^3}\Bigg[\Bigg\{-\frac{4p^2}{\Lambda^2}\frac{\partial F(P)}{\partial P_\mu} - 4F(P)\frac{p_\mu}{\Lambda^2}\Bigg\}G(P)\mathcal{K}\left(\frac{p^2}{\Lambda^2}\right)\mathcal{K}'\left(\frac{p^2}{\Lambda^2}\right) - \frac{4p^2p_\mu}{\Lambda^4}F(P)G(P)\Bigg\{\mathcal{K}\left(\frac{p^2}{\Lambda^2}\right)\mathcal{K}''\left(\frac{p^2}{\Lambda^2}\right) + \mathcal{K}'^2\left(\frac{p^2}{\Lambda^2}\right)\Bigg\}\Bigg]. \tag{3.13}$$

Now we convert to cylindrical coordinates in space-time by writing $P_\mu = y\Lambda(v_F x, \cos\theta, \sin\theta)$ and integrate over y, x, and θ. Also, let us define $\hat{P}_\mu = (v_F x, \cos\theta, \sin\theta)$ and $\hat{p}_\mu = (0,\cos\theta,\sin\theta)$. We use the homogeneity properties of the functions F and G to obtain

$$\Lambda\frac{d}{d\Lambda}\Sigma_1(K) \approx \frac{v_F K_\mu}{8\pi^3}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \Bigg[\Bigg\{-4\frac{\partial F(\hat{P})}{\partial P_\mu} - 4\hat{p}_\mu F(\hat{P})\Bigg\}G(\hat{P})\int_0^\infty y\,dy\,\mathcal{K}(y^2)\mathcal{K}'(y^2) - 4\hat{p}_\mu F(\hat{P})G(\hat{P})\int_0^\infty y^3dy\{\mathcal{K}(y^2)\mathcal{K}''(y^2) + \mathcal{K}'^2(y^2)\}\Bigg]. \tag{3.14}$$

Now the integrals over y can be evaluated exactly by integration by parts, and the final result is *independent* of the precise form of $\mathcal{K}(y)$,

$$\Lambda\frac{d}{d\Lambda}\Sigma_1(K) = \frac{v_F K_\mu}{8\pi^3}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \frac{\partial F(\hat{P})}{\partial P_\mu} G(\hat{P}). \tag{3.15}$$

In a similar manner, we can write the result for the vertex Ξ in Eq. (2.17) in the schematic form,

$$\Xi = \tau^x + \int \frac{d^3P}{8\pi^3} H(P)\mathcal{K}^3\left(\frac{p^2}{\Lambda^2}\right), \tag{3.16}$$

where $H(P)$ is a homogenous function of P of degree of -3. Proceeding as above we obtain the analog of Eq. (3.15),

$$\Lambda\frac{d}{d\Lambda}\Xi = \frac{v_F}{8\pi^3}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta H(\hat{P}). \tag{3.17}$$

We can now use the above expressions to obtain explicit results for the constants C_{1-4} at order $1/N_f$. For C_{1-3} we combine Eqs. (2.16), (3.3), (3.11), and (3.15) to obtain

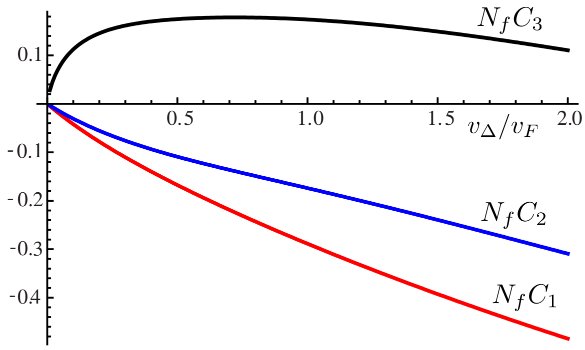

FIG. 4. (Color online) Plots of the functions C_{1-3} in Eq. (3.18).

$$C_1 = \frac{2(v_\Delta/v_F)}{\pi^3 N_f}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \times \frac{[x^2-\cos^2\theta-(v_\Delta/v_F)^2\sin^2\theta]}{[x^2+\cos^2\theta+(v_\Delta/v_F)^2\sin^2\theta]^2}\mathcal{G}(x,\theta),$$

$$C_2 = \frac{2(v_\Delta/v_F)}{\pi^3 N_f}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \times \frac{[-x^2+\cos^2\theta-(v_\Delta/v_F)^2\sin^2\theta]}{[x^2+\cos^2\theta+(v_\Delta/v_F)^2\sin^2\theta]^2}\mathcal{G}(x,\theta),$$

$$C_3 = \frac{2(v_\Delta/v_F)}{\pi^3 N_f}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta \times \frac{[x^2+\cos^2\theta-(v_\Delta/v_F)^2\sin^2\theta]}{[x^2+\cos^2\theta+(v_\Delta/v_F)^2\sin^2\theta]^2}\mathcal{G}(x,\theta), \quad (3.18)$$

where

$$\mathcal{G}^{-1}(x,\theta) = \frac{x^2+\sin^2\theta}{\sqrt{x^2+(v_\Delta/v_F)^2\cos^2\theta+\sin^2\theta}} + \frac{x^2+\cos^2\theta}{\sqrt{x^2+\cos^2\theta+(v_\Delta/v_F)^2\sin^2\theta}} \quad (3.19)$$

is the ϕ propagator inverse. All the integrals in Eq. (3.18) are well defined and convergent for all nonzero v_Δ/v_F and have to be evaluated numerically. It is also evident that the results are functions only of v_Δ/v_F. The results of a numerical evaluation are shown in Fig. 4.

Similarly, we can combine Eqs. (2.17), (3.3), (3.16), and (3.17) to deduce that

$$C_4 = -C_3 \quad (3.20)$$

at leading order in the $1/N_f$ expansion. With the values in Eq. (3.18), we are now in a position to numerically solve the RG flow for the velocity ratio v_Δ/v_F in Eq. (3.10). We plot the right-hand side of this flow equation in Fig. 5. Note that the only zero of the flow equation is at $v_\Delta/v_F=0$, and that this fixed point is attractive in the infrared. So for all starting values of the parameter v_Δ/v_F, the flow is toward $v_\Delta/v_F \to 0$ for large ℓ. At large enough scales, it is always possible to approximate the constants C_{1-3} in Eq. (3.18) by their limiting values for small v_Δ/v_F. Evaluating the integrals in Eq. (3.18) in this limit, we obtain

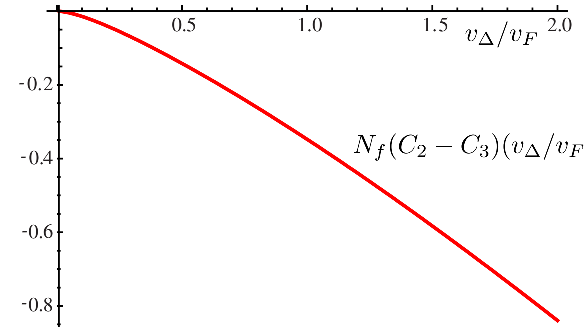

FIG. 5. (Color online) Right-hand side of the RG flow equation [Eq. (3.10)] for v_Δ/v_F.

$$C_1 = -0.4627\frac{(v_\Delta/v_F)}{N_f} + \mathcal{O}[(v_\Delta/v_F)^3],$$

$$C_2 = -0.3479\frac{(v_\Delta/v_F)}{N_f} + \mathcal{O}[(v_\Delta/v_F)^3],$$

$$C_3 = \left(\frac{8}{\pi^2}\ln(v_F/v_\Delta) - 0.9601\right)\frac{(v_\Delta/v_F)}{N_f} + \mathcal{O}[(v_\Delta/v_F)^3]. \quad (3.21)$$

The logarithmic dependence on v_Δ/v_F arises from a singularity in the integrand at $x=0$ and $\theta=\pm\pi/2$: this corresponds to an integral across the underlying Fermi surface over ω and k_x where the inverse fermion propagator $\sim -i\omega + v_F k_x \tau^z$. Inserting these values into the flow equations [Eqs. (3.8)–(3.10)], we obtain an explicit form of the RG flow for small v_Δ/v_F,

$$\frac{dv_\Delta}{d\ell} = -\left(\frac{8}{\pi^2}\ln(v_F/v_\Delta) - 0.4974\right)\frac{v_\Delta^2}{N_f v_F}, \quad (3.22)$$

$$\frac{dv_F}{d\ell} = -0.1148\frac{v_\Delta}{N_f}, \quad (3.23)$$

$$\frac{d(v_\Delta/v_F)}{d\ell} = -\left(\frac{8}{\pi^2}\ln(v_F/v_\Delta) - 0.6122\right)\frac{(v_\Delta/v_F)^2}{N_f}. \quad (3.24)$$

The right-hand side of Eq. (3.24) has a zero for $v_\Delta/v_F \approx 2$; this is spurious as this equation is valid only for $v_\Delta/v_F \ll 1$. The full expression for the right-hand side, valid for arbitrary v_Δ/v_F, is plotted in Fig. 5, and this makes it clear that the only zero of the beta function is at $v_\Delta/v_F=0$.

The results of a numerical integration of Eqs. (3.8)–(3.10) are shown in Figs. 6–8. We start the RG integration at $\ell=0$ corresponding to a temperature $T=T_0$ at the nematic critical point, at which point the velocities have the values v_F^0 and v_Δ^0. Integrating to the scale ℓ yields the velocities $v_F(\ell)$ and $v_\Delta(\ell)$ at a temperature $T=T_0 e^{-\ell}$. The results depend on the initial value of the velocity ratio, v_Δ^0/v_F^0, and are shown in the

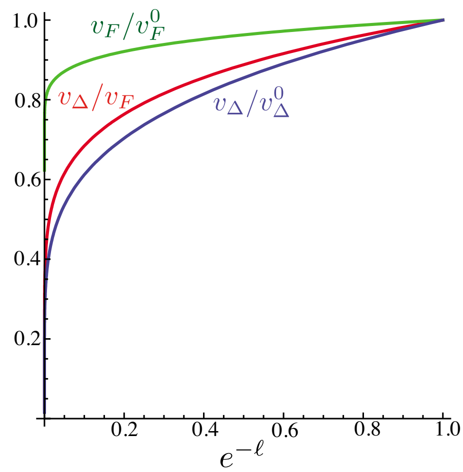

FIG. 6. (Color online) Integration of Eqs. (3.8)–(3.10) from ≪ $=0$ to ℓ for $N_f=2$, starting from the velocities v_F^0 and v_Δ^0. Results above are for $v_\Delta^0/v_F^0=1$.

figures for $v_\Delta^0/v_F^0=1$, 0.1, and 0.05. These results can also be converted into dependence on the deviation from the nematic critical point, r, by integrating Eq. (3.6).

In the asymptotic low-temperature regime ($\ell \to \infty$), we have $v_\Delta/v_F \to 0$, and so we can integrate the asymptotic expression in Eq. (3.24). This yields

$$\frac{v_\Delta}{v_F} = \frac{\pi^2 N_f}{8} \frac{1}{\ell} \frac{1}{\ln[0.3809\ell/N_f]}. \tag{3.25}$$

The T dependence follows by using $\ell = \ln(T_0/T)$. Using the result for v_Δ/v_F in Eq. (3.25) and integrating Eq. (3.23), we find that v_F only has a finite renormalization from its bare value as $\ell \to \infty$. So the decrease in v_Δ/v_F as $\ell \to \infty$ (i.e., as $T \to 0$) comes primarily from the decrease in v_Δ. This is also clear from the plots in Figs. 6–8.

We can also apply these results to obtain the $T=0$ evolution of the velocities as a function of the tuning parameter, r, across the nematic ordering transition. In this case we use Eq. (3.6), along with the fact that $\nu \to 1$ as $v_\Delta/v_F \to 0$. Then we deduce that $\ell = \ln(1/|r|)$ to leading logarithm accuracy,

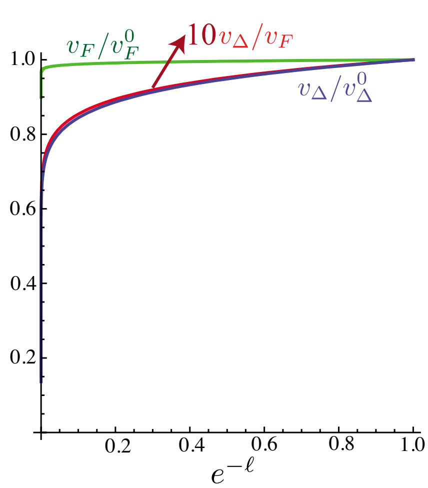

FIG. 7. (Color online) As in Fig. 6 but for $v_\Delta^0/v_F^0=0.1$.

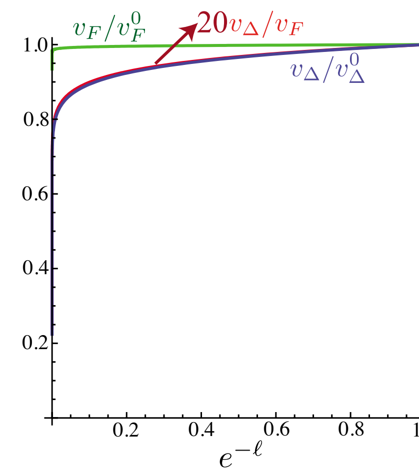

FIG. 8. (Color online) As in Fig. 6 but for $v_\Delta^0/v_F^0=0.05$.

and so obtain the r dependence of the velocities from Figs. 6–8. When both T and r are nonzero, we can use $\ell = \ln(\max(|r|, T))$ to leading logarithmic accuracy. Note that this predicts a minimum in v_Δ/v_F as r is tuned across the nematic ordering transition with the minimum value of order $1/\ln(1/T)$.

IV. THEORY FOR SMALL v_Δ/v_F

The RG analysis in Sec. III has shown that the β functions at order $1/N_f$ are such that the flow is toward the strong anisotropy limit, $v_\Delta/v_F \to 0$, at large scales. Here we will show that this conclusion holds at all orders in the $1/N_f$ expansion. Indeed, for small v_Δ/v_F we will show that v_Δ/v_F can itself be as the control parameter for the computation, and N_f is not required to be large. In other words, the RG flow equations in Eqs. (3.22)–(3.24) are asymptotically *exact*, even for the physical value of $N_f=2$. All corrections to Eqs. (3.22)–(3.24) are higher order in v_Δ/v_F.

Our conclusions follow from an examination of the structure of the action S_ϕ for the nematic order in Eq. (2.6) in the limit of $v_\Delta/v_F \to 0$. As we will shortly verify, the natural scale for the fluctuations of ϕ are at frequency and momenta with $\omega \sim v_F k_x \sim v_F k_y$. Under these conditions, we can just take the $v_\Delta \to 0$ limit of the expressions in Eqs. (2.8), (2.10), and (2.12). This limit exists, and to quadratic order in ϕ we have the following effective action as $v_\Delta \to 0$ (at criticality, $r=0$):

$$S_\phi = \frac{N_f}{16 v_F v_\Delta} \int \frac{d^2k}{4\pi^2} \int \frac{d\omega}{2\pi} |\phi(k,\omega)|^2 [\sqrt{\omega^2 + v_F^2 k_x^2} + \sqrt{\omega^2 + v_F^2 k_y^2}] + \cdots . \tag{4.1}$$

Rescaling momenta $k \to k/v_F$, we notice that the prefactor in Eq. (4.1) is proportional to the dimensionless number $N_f v_F/v_\Delta$. Thus each propagator of ϕ will appear with the small prefactor of $v_\Delta/(N_f v_F)$. Further, as claimed above, each propagator has the typical scales $\omega \sim v_F k_x \sim v_F k_y$.

These considerations are easily extended to terms to all orders in ϕ in Eq. (2.6). The full expression for S_ϕ involves the sum of the logarithms of the determinants of the Dirac operators of $\Psi_{1,2}$ fermions in a background ϕ field. The expansion of these determinants in powers of ϕ involves only terms with a single fermion loop. For each Ψ_1 loop we rescale the *fermion* momentum $p_x \to p_x/v_F$ and $p_y \to p_y/v_\Delta$. Similarly for each Ψ_2 loop we rescale the fermion momentum $p_x \to p_x/v_\Delta$ and $p_y \to p_y/v_F$. In both cases, the integral over the fermion loop momentum yields a prefactor of $1/(v_F v_\Delta)$. For the ϕ vertices emerging from the Ψ_1 loops, we have momenta $\omega \sim v_F k_x \sim v_F k_y$, as noted above; when this external momenta enters the fermion loop, we have $k_x \sim p_x$, the typical Ψ_1 x momentum. However, $k_y \ll p_y \sim 1/v_\Delta$, the typical Ψ_1 y momentum. Consequently, we may safely set $k_y=0$ within the fermion loop, and there is no further dependence on v_Δ in the loop integral. Similarly, in the Ψ_2 loops, we may set $k_x=0$ within the fermion loop. To summarize, the typical ϕ momenta are $k_x \sim 1/v_F$ and $k_y \sim 1/v_F$, the typical Ψ_1 momenta are $p_x \sim 1/v_F$ and $p_y \sim 1/v_\Delta$, and the typical Ψ_2 momenta are $p_x \sim 1/v_\Delta$ and $p_y \sim 1/v_F$. Further, the resulting effective action for ϕ has the form

$$S_\phi = \frac{N_f}{v_F v_\Delta} \mathcal{S}[\phi(k,\omega); v_F k_x, v_F k_y, \omega] + \text{terms higher order in } v_\Delta/v_F. \tag{4.2}$$

Here $\mathcal{S}$ is a functional of ϕ which is independent of v_Δ, whose expansion in ϕ has coefficients which depend only on $v_F k_x$ and $v_F k_y$, and ω. It can be verified that Eq. (4.1) and the quartic terms in Eqs. (2.10) and (2.12) are indeed of the form in Eq. (4.2). From Eq. (4.2), it is evident (after rescaling momenta $k \to k/v_F$ and momentum space fields $\phi \to v_F^2 \phi$) that the natural expansion parameter controlling ϕ fluctuations is $v_\Delta/(N v_F)$. This is the main result of this section.

It remains understanding the $\ln(v_F/v_\Delta)$ factor in Eq. (3.24). If we compute observable ϕ correlations under the reduced effective action $\mathcal{S}$, the resulting expansion in powers of $v_\Delta/(N v_F)$ leads to Feynman integrals which are not necessarily infrared or ultraviolet finite. Because the action $\mathcal{S}$ is free of any dimensionful coupling constant, the Feynman graph divergences can at most be powers of logarithms. These divergences are cutoff by using the full fermion propagators, including the terms proportional to v_Δ times the momenta of the boson propagators. This leads to factors of $\ln(v_F/v_\Delta)$, as was the case in obtaining Eq. (3.24) from Eq. (3.18).

V. CONCLUSIONS

This paper has described the RG properties of the field theory[1] in Eq. (2.5) for nematic ordering. It was previously noted[7] that the effective potential for the nematic order, ϕ, remained unrenormalized upon integrating out the nodal fermions $\Psi_{1,2}$. This happens because the ϕ couples to a conserved fermion current, and a space-time-independent ϕ can be "gauged away" applying a gauge transformation to the fermions. One consequence of this nonrenormalization is that the susceptibility exponent, γ, takes the simple value in Eq. (2.14). However, nontrivial renormalizations of the field scale of both ϕ and $\Psi_{1,2}$ are still possible, along with renormalizations of the velocities, and we have shown here that this leads to an interesting RG flow structure with non-mean-field exponents.

It is interesting to note here the similarities to the RG flow of supersymmetric field theories.[11] There the effective potential also remains unrenormalized (albeit for very different reasons), but the "wave-function renormalizations" lead to many nontrivial RG fixed points. Our main result was that the transition is described by a fixed point in which the fermion velocities at the nodal points have a ratio which approaches a fixed point with $v_\Delta/v_F \to 0$ logarithmically slowly. However, it is not valid to set the pairing-induced velocity $v_\Delta=0$ in the computation, and so deal with a metallic Fermi surface. For the case where the fermion dispersion is $\sim(v_F^2 k_x^2 + v_\Delta^2 k_y^2)^{1/2}$, the typical fermion momenta contributing to the critical theory scale as $k_x \sim 1/v_F$ and $k_y \sim 1/v_\Delta$, and so the full form of the Bogoliubov quasiparticle dispersion is important. The flow of the velocities is described by Eqs. (3.22)–(3.24) for $v_\Delta/v_F \ll 1$, and these equations are believed to be asymptotically exact.

Unlike the fermions, the fluctuations of the nematic order, ϕ, are isotropic in momenta. These are described asymptotically exactly by Eq. (4.1). Note that the propagator is very different from free field, and has a large anomalous dimension, as was found in the $N_f=\infty$ theory.[7] Experimental detection of this unusual spectrum in, e.g., inelastic x-ray scattering would be most interesting.

ACKNOWLEDGMENTS

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